Delta-matroid polynomials and the symmetric Tutte polynomial

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Motivation

Definition (Aigner & van der Holst 2004, and Bouchet 1991)

The (single-variable) interlace polynomial of a graph $G$ is

$$q(G; y) = \sum_{X \subseteq V(G)} (y - 1)^{n(A(G)[X])}.$$

Theorem (Aigner & van der Holst 2004, and Bouchet 1991)

Let $M$ be a binary matroid and $G$ be the fundamental bipartite graph of $M$ with respect to some basis $B$. Then $T(M; y, y) = q(G; y)$.

- $T(M; y, y)$ is defined for arbitrary matroids (instead of only binary matroids).
- $q(G; y)$ is defined for arbitrary graphs (instead of only bipartite graphs).
- Goal: common generalization for $T(M; y, y)$ and $q(G; y)$. 

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Delta-matroid polynomials
### Summary

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$\Delta$-matroids [Bouchet 1988] generalize both adjacency matrices and matroids.

Key: $q(D; y)$ and $Q(D; y)$ retain many of the attractive properties: recursive relations, various evaluations, etc.
Let $\Delta$ be symmetric difference.

**Definition (Bouchet 1988)**

A nonempty set system $D = (V, B)$ is a $\Delta$-matroid over $V$ if for all $X, Y \in B$ and $u \in X \Delta Y$, there is an element $v \in X \Delta Y$ such that $X \Delta \{u, v\} \in B$ (we allow $u = v$).

**Theorem (Bouchet 1988)**

A set system is a matroid (described by its bases) iff it is an equicardinal $\Delta$-matroid.
\textbf{Definition}

Twist of $D$ on $X \subseteq V$ is $D \ast X := (V, B \ast X)$ where $B \ast X = \{Y \bigtriangleup X \mid Y \in D\}$.

Twist generalizes matroid duality: $M \ast V = M^*$.

\textbf{Theorem (Bouchet 1988)}

$D$ is a $\Delta$-matroid iff $D \ast X$ is a $\Delta$-matroid.

$\Delta$-matroids have deletion and contraction, generalizing deletion and contraction for matroids.

\textbf{Theorem}

A set system $D$ is a $\Delta$-matroid iff $\min(D \ast X)$ is equicardinal for all $X \subseteq V$. 
Representable $\Delta$-matroids

**Theorem (Bouchet 1988)**

For a skew-symmetric $V \times V$-matrix $A$ (over a field $\mathbb{F}$),
$D_A := (V, B_A)$ with $B_A = \{X \subseteq V \mid A[X] \text{ nonsingular}\}$ is a $\Delta$-matroid.

**Definition**

A $\Delta$-matroid $D$ over $V$ is *representable over* $\mathbb{F}$ if $D = D_A * X$ for a $V \times V$-skew-symmetric $A$ over $\mathbb{F}$ with $X \subseteq V$.

**Theorem (Bouchet 1988)**

A matroid is representable over $\mathbb{F}$ in the usual matroid sense iff it is representable over $\mathbb{F}$ in this $\Delta$-matroid sense.

A $\Delta$-matroid $D$ is called *binary* if representable over $GF(2)$. 
interlace polynomial as $\Delta$-matroid polynomial

**Definition**

Let $D = (V, B)$ be a $\Delta$-matroid. Define $d_D$ as the common cardinality of the elements of $\min(B)$.

So, $d_{D^*X}$ is Hamming distance of $X$ from $D$.

**Theorem**

For any graph $G$, $d_{D_{A(G)^*X}} = n(A(G)[X])$.

**Corollary**

For any graph $G$,

$$q(G; y) = \sum_{X \subseteq V(G)} (y - 1)^{n(A(G)[X])} = \sum_{X \subseteq V(G)} (y - 1)^{d_{D_{A(G)^*X}}}.$$
Corollary

For any graph $G$,

$$q(G; y) = \sum_{X \subseteq V(G)} (y - 1)^{d_{D_{A(G)}^*}X}.$$ 

Generalization from binary $\Delta$-matroids to arbitrary $\Delta$-matroids:

Definition ($\Delta$-matroid polynomial)

Let $D$ be a $\Delta$-matroid over $V$.

$$q(D; y) := \sum_{X \subseteq V} (y - 1)^{d_{D^*}X}.$$ 

So, $q(G; y) = q(D_{A(G)}; y)$. 

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Delta-matroid polynomials
Definition (Tutte polynomial)

Let $M$ be a matroid over $V$.

$$T(M; x, y) := \sum_{X \subseteq V} (x - 1)^{n_{M^*}(V \setminus X)} (y - 1)^{n_M(X)}.$$ 

Recall that $n_M(X) = |X| - r_M(X)$.

$$n_{M^*}(V \setminus X) + n_M(X) = d_{M^*X}$$

Theorem

Let $M$ be a matroid over $V$

$$T(M; y, y) = \sum_{X \subseteq V} (y - 1)^{d_{M^*X}} = q(M; y).$$
Loop and coloop compatible with matroids.

**Definition**

Let $D = (V, B)$ be a $\Delta$-matroid. $v \in V$ is

- **loop** if for all $X \in B$, $v \notin X$,
- **coloop** if $D \ast v$ is loop,
- **singular** if $v$ is either loop or coloop.

Deletion and contraction compatible with matroids.

**Definition (deletion)**

Let $D = (V, B)$ be a $\Delta$-matroid and $v \in V$. If $v$ is not a coloop, then $D \setminus v := (V \setminus \{v\}, B')$ with $B' = \{X \in B \mid v \notin X\}$. If $v$ is a coloop, then $D \setminus v := D \ast v \setminus v$.

Contraction: $D \ast v \setminus v$. 

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Delta-matroid polynomials
Theorem

Let $D$ be a $\Delta$-matroid over $V$. If $V = \emptyset$, then $q(D; y) = 1$.

If $v \in V$ is nonsingular in $D$, then

\[ q(D; y) = q(D \setminus v; y) + q(D \ast v \setminus v; y). \]

If $v \in V$ is singular in $D$, then

\[ q(D; y) = yq(D \setminus v; y) = yq(D \ast v \setminus v; y). \]

Two types of minor operations: deletion and contractions.
For a graph $G$ and $Y \subseteq V(G)$. Let $G + Y$ be the graph obtained from $G$ by toggling the existence of loops for the vertices of $Y$.

**Definition (Aigner & van der Holst 2004, and Bouchet 1991)**

Let $G$ be a graph. Then the *global interlace polynomial* of $G$ is

$$Q(G; y) = \sum_{X \subseteq V(G)} \sum_{Y \subseteq X} (y - 2)^{n(A(G+Y)[X])}.$$
Definition

Let $D = (V, B)$ be a $\Delta$-matroid (or, more generally, set system) and $X \subseteq V$. Define loop complementation of $D$ on $X$ by $D + X = (V, B')$ where $Y \in B'$ iff $|\{Z \in B \mid Y \setminus X \subseteq Z \subseteq Y\}|$ is odd.

$D + X$ not necessarily a $\Delta$-matroid.

Theorem

Let $A$ be a symmetric $V \times V$-matrix and $X \subseteq V$. Then $\mathcal{D}_{A+X} = \mathcal{D}_A + X$.

The class of binary $\Delta$-matroids is closed under $\plus$. Extendable to $GF(4)$. 
Let $D$ be a $\Delta$-matroid (or, more generally, set system). Then $(D + X) + X = D$. In fact, $+X$ and $\ast X$ are involutions that generate $S_3$ and commutes on disjoint sets.

Third involution: $D\ast X := D + X \ast X + X = D \ast X + X \ast X$. 
Let $\mathcal{P}_3(V)$ be the set of ordered 3-partitions of $V$.

**Definition**

Let $D$ be a $\Delta$-matroid. Define

$$Q(D; y) = \sum_{(A, B, C) \in \mathcal{P}_3(V)} (y - 2)^{d_D * B * C}.$$

**Theorem**

Let $G$ be a graph. Then $Q(G; y) = Q(D_G; y)$. 

In general, a \( \Delta \)-matroid \( D \) is \textit{vf-safe} if applying any sequence of twist and loop complementation obtains a \( \Delta \)-matroid.

\( v \in V \) is \textit{strongly nonsingular} if \( v \) is nonsingular and \( D \ast v \neq D \).

\begin{align*}
\text{Theorem} \\
\text{Let } D \text{ be a vf-safe } \Delta \text{-matroid and let } v \in V. \\
\begin{enumerate}
\item If \( v \) is strongly nonsingular in \( D \), then
\[ Q(D; y) = Q(D \setminus v; y) + Q(D \ast v \setminus v; y) + Q(D\bar{\ast}v \setminus v; y). \]
\item If \( v \) is not strongly nonsingular in \( D \), then
\[ Q(D; y) = yQ(D \setminus v; y). \]
\end{enumerate}
\end{align*}

Three types of minor operations!
Let $D$ be a $\Delta$-matroid.

1. If $D$ is even and $|V| > 0$, then $q(D; 0) = 0$.
2. If $D$ is vf-safe, then $q(D; -1) = (-1)^{|V|}(-2)^{d_{D^*}V}$ (third direction!).
3. If $D$ is vf-safe with $|V| > 0$, then $Q(D; 0) = 0$.
4. If $D$ is binary, then $q(D)(3) = k |q(D)(-1)|$ for some odd integer $k$ [Bouchet].
Penrose polynomial

**Definition**

Let $D$ be a vf-safe $\Delta$-matroid. The Penrose polynomial of $D$ is

$$P(D; y) = \sum_{X \subseteq V} (-1)^{|X|} y^{d_{D*V}X}.$$ 

Recursive relation is outside realm of matroids.

**Theorem**

Let $D$ be a vf-safe $\Delta$-matroid. If $V = \emptyset$, then $P_M(y) = 1$.

If $v \in V$ is

- nonsingular in $D*V$, then
  $$P(D; y) = P(D * v \setminus v; y) - P(D*V \setminus v; y),$$
- a coloop of $D*V$, then
  $$P(D; y) = (1 - y)P(D * v \setminus v; y),$$
- a loop of $D*V$, then
  $$P(D; y) = (y - 1)P(D*V \setminus v; y).$$

Multivariate version to incorporate all these $\Delta$-matroid polynomials.
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Thanks!