

**NOTES ON DE JONG'S PERIOD=INDEX THEOREM FOR  
CENTRAL SIMPLE ALGEBRAS OVER FIELDS OF  
TRANSCENDENCE DEGREE TWO**

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1. INTRODUCTION

These are notes on de Jong's proof of the period=index theorem over fields of transcendence degree two. They are actually about the "simplified" proof sketched by de Jong in the last section of his paper. These notes were meant as support for my lectures at the summer school "Central Simple Algebras over Function Fields of Surfaces" at the Universität Konstanz between August, 26 and September, 1 2007 but I did not finish them in time.<sup>1</sup>

No originality is intended here. I have copied mostly from the following sources

- M. Artin and A. J. de Jong, *Stable orders over surfaces*, manuscript 2004, 198 pages.
- J.-L. Colliot-Thélène, *Algèbres simples centrales sur les corps de fonctions de deux variables (d'après A. J. de Jong)*, Astérisque (2006), no. 307, Exp. No. 949, ix, 379–413, Séminaire Bourbaki. Vol. 2004/2005.
- A. J. de Jong, *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), no. 1, 71–94.
- M. Ojanguren and R. Parimala, *Smooth finite splittings of Azumaya algebras over surfaces*. <http://www.mathematik.uni-bielefeld.de/lag/man/244.ps.gz>.

I wish to thank Johan de Jong for answering some questions and for sending me his manuscript with Mike Artin on stable orders.

**Disclaimer and apology** These notes are not very systematic. In particular there are some inconsistencies of notation and the level of detail is very uneven. Several parts are directly copied from the references listed above. Some parts consider subjects covered by other lecturers. Since these notes were mostly written before the summer school my presentation here is often different in content and notation from the actual lectures during the summer school. I apologize for this.

**Notations and conventions** Unless otherwise specified all cohomology is étale cohomology,  $k$  is an algebraically closed field and  $n$  is a number which is prime to the characteristic.

As over fields we define the degree of an Azumaya algebra as the square root of its rank (assuming it has constant rank).

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<sup>1</sup>The other lecturers were Philippe Gille, Andrew Kresch, Max Lieblich, Tamás Szamuely and Jan Van Geel.

## 2. DE JONG'S THEOREM

Here is de Jong's celebrated result.

**Theorem 2.1.** [8] *If  $K$  is a field of transcendence degree 2 and if  $A$  is a central simple algebra over  $K$  whose Brauer class has order prime to the characteristic then*

$$\text{index}(D) = \text{period}(D)$$

We start with some observations and comments.

- (1) Max Lieblich [16] has proved that the period-index problem over function fields can be lifted to characteristic zero. Hence the condition that the period must be prime to the characteristic can be removed. However in these notes we stick to de Jong's original setup.
- (2) It is easy to reduce the period-index problem to the case that  $K$  is finitely generated.
- (3) If  $K$  is finitely generated then by resolution of singularities we may assume that  $K$  is the function field of a smooth projective connected surface  $X$  (notation:  $K = k(X)$ ).
- (4) If  $K = k(X)$  as above then one may first consider the case that  $D$  is in the image of  $\text{Br}(X)$ . This is the so-called "unramified case". De Jong's proof deals first with the unramified case. The proof uses a deformation argument which is very specific for surfaces.
- (5) The "ramified" case in de Jong's proof is treated by an extremely ingenious reduction to the unramified case.
- (6) In [16] Max Lieblich gives a new and completely different proof of de Jong's theorem which starts by (birationally) fibering the surface  $X$  over  $\mathbb{P}^1$ . He then rephrases the period-index problem as an existence problem for certain twisted vector bundles on curves over  $k(t)$ . The corresponding moduli space is (geometrically) rationally connected (rational even) and in this way one can recover the period-index theorem from (deep) results about the existence of rational points on rationally connected varieties. See [10].
- (7) Yet another proof is by Star and de Jong [9]. Here one reduces the problem to the existence of rational points in certain twisted Grassmanians over the function field of a surface.

## 3. SPECIALIZATION

De Jong's proof uses several times a specialization argument for the index. We discuss this first.

**Theorem 3.1.** *Assume that  $L/K$  is a field extension and assume that there is discrete valuation ring in  $L$  which contains  $K$ , whose quotient field is  $L$  and whose residue field is  $K$ . Let  $D$  be a central division algebra over  $K$ . Then*

$$\text{index}(L \otimes_K D) = \text{index}(D)$$

*Proof.* Let  $m$  be the maximal ideal of  $R$ . Put  $A = R \otimes_K D$  and filter  $A$  by the  $m$ -adic filtration. Then  $\text{gr } A = (\text{gr } R) \otimes_K D = D[t]$ . Hence  $\text{gr } A$  is a domain and thus so is  $L \otimes_K D = L \otimes_R A$ . Thus  $L \otimes_K D$  is a division algebra and therefore

$$\text{index}(L \otimes_K D) = \text{deg}(L \otimes_K D) = \text{deg}(D) = \text{index}(D)$$

finishing the proof. □

*Remark 3.2.* It is easy to see that this theorem and its proof generalize to the case that  $R$  is regular local (gr  $A$  becomes a polynomial ring in several variables). However the theorem is false without the assumption of  $R$  being regular. Consider the following counter example. Let  $D$  be arbitrary (of index  $> 1$ ) and let  $X$  be the Brauer-Severi variety of  $D$ . It is well-known that  $X$  has a very ample line bundle  $\mathcal{L}$ . Let  $R$  be the local ring of the corresponding cone at the singular vertex. Then the quotient field  $L$  of  $R$  is  $k(X)[t]$ . Hence  $L$  splits  $D$  and thus  $\text{index}(L \otimes_K D) = 1 < \text{index}(D)$ .

#### 4. INVARIANTS

Let  $X$  be a scheme. It is classical that

$$H^1(X, \mathbb{G}_m) = \text{Pic}(X)$$

The *cohomological Brauer group* of  $X$  is defined as  $H^2(X, \mathbb{G}_m)_{\text{tors}}$ . Thanks to the following theorem we know that the cohomological Brauer group often coincides with the usual Brauer group.

**Theorem 4.1.** (*Gabber*) *Assume that  $X$  is a quasi-compact, separated scheme with an ample invertible sheaf (e.g.  $X$  is quasi-projective). Then*

$$H^2(X, \mathbb{G}_m)_{\text{tors}} = \text{Br}(X)$$

Gabber's proof of this result is not widely disseminated. However a different proof was given by de Jong. See [7].

Let  $\mu_n$  denote the  $n$ -th roots of unity. Since by our standing hypotheses  $n$  is invertible we have an exact sequence for the étale topology

$$1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \rightarrow 1$$

(this is the Kummer sequence). Taking cohomology we get a short exact sequence

$$(4.1) \quad 0 \rightarrow \text{Pic}(X)/n \text{Pic}(X) \rightarrow H^2(X, \mu_n) \xrightarrow{[-]} \text{Br}(X)_n \rightarrow 0$$

We find that the cohomology group  $H^2(X, \mu_n)$  is a finer invariant than  $\text{Br}(X)_n$ .

Now we look at

$$1 \rightarrow \mu_n \rightarrow \text{Sl}_n \rightarrow \text{PGL}_n \rightarrow 1$$

Taking into account that [18, Proof of Thm IV.2.5]

$$H^1(X, \text{PGL}_n) = \text{Az}_n(X)$$

where  $\text{Az}_n(X)$  denotes the isomorphism classes of Azumaya algebras of rank  $n^2$  we obtain a map

$$\text{cl}(-) : \text{Az}_n(X) \rightarrow H^2(X, \mu_n)$$

Thus we have constructed maps

$$\text{Az}_n(X) \xrightarrow{\text{cl}(-)} H^2(X, \mu_n) \xrightarrow{[-]} \text{Br}(X)_n$$

We often denote the composition by  $[-]$  as well.

**Lemma 4.2.** *The square in the following diagram is commutative up to sign*

$$\begin{array}{ccccc}
\mathrm{Az}_n(X) & \xrightarrow{\mathrm{cl}(-)} & H^2(X, \mu_n) & \xrightarrow{[-]} & \mathrm{Br}(X)_n \\
\mathrm{End}_{\mathcal{O}_X}(-) \uparrow & & \uparrow & & \\
\mathrm{Vect}_n(X) & \xrightarrow{\wedge^n(-) \bmod n} & \mathrm{Pic}(X)/n \mathrm{Pic}(X) & & 
\end{array}$$

where  $\mathrm{Vect}_n(X)$  denotes the isoclasses of vector bundles of rank  $n$  on  $X$ .

*Proof.* This follows from the following commutative diagram of groups

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mu_n & \longrightarrow & \mathrm{Sl}_n & \longrightarrow & \mathrm{PGL}_n \longrightarrow 1 \\
& & \mathrm{id} \uparrow & & \uparrow \mathrm{pr}_1 & & \uparrow \\
1 & \longrightarrow & \mu_n & \xrightarrow{x \mapsto (x, x^{-1})} & \mathrm{Sl}_n \times \mathbb{G}_m & \xrightarrow{(u, v) \mapsto uv} & \mathrm{GL}_n \longrightarrow 1 \\
& & (-)^{-1} \downarrow & & \downarrow \mathrm{pr}_2 & & \downarrow \det \\
1 & \longrightarrow & \mu_n & \longrightarrow & \mathbb{G}_m & \xrightarrow{(-)^n} & \mathbb{G}_n \longrightarrow 1
\end{array}$$

Taking into account that

$$H^1(X, \mathrm{GL}_n) = \mathrm{Vect}_n(X)$$

and that the map

$$H^1(X, \mathrm{GL}_n) \rightarrow H^1(X, \mathrm{PGL}_n)$$

corresponds to

$$E \mapsto \mathrm{End}_{\mathcal{O}_X}(E)$$

(see [18, Proof of Thm IV.2.5]) we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Az}_n(X) & \xrightarrow{\mathrm{cl}} & H^2(X, \mu_n) \\
\mathrm{End}_{\mathcal{O}_X}(-) \uparrow & & \parallel \\
\mathrm{Vect}_n(X) & \longrightarrow & H^2(X, \mu_n) \\
\wedge^n(-) \downarrow & & \downarrow \times(-1) \\
\mathrm{Pic}(X) & \longrightarrow & H^2(X, \mu_n)
\end{array}$$

which yields what we want.  $\square$

## 5. RAMIFICATION

It is well known that étale cohomology can be computed in terms of local invariants. We discuss this in a very special case.

Let  $l$  be a prime number different from the characteristic.<sup>2</sup> Let  $\mu_l$  denote the  $l$ 'th roots of unity. The following is the version by Bloch and Ogus of the Grothendieck coniveau spectral sequence [13, p164].

<sup>2</sup>Most of what we say is valid for  $l$  not divisible by the characteristic.

**Theorem 5.1.** [5, Prop (3.9)] *Assume that  $X$  is smooth over a perfect ground field  $k$ . Then there is a spectral sequence*

$$(5.1) \quad E_1^{pq} = \bigoplus_{x \in X^{(p)}} H^{q-p}(x, \mu_l^{\otimes n-p}) \Rightarrow H^{p+q}(X, \mu_l^{\otimes n})$$

Here  $X^{(p)}$  are the points of codimension  $p$  in  $X$ . I.e. the points such that  $\mathcal{O}_{X,x}$  has (Krull) dimension  $p$ .

Furthermore  $H^p(x, \mu_l^{\otimes n})$  is defined as

$$H^p(x, \mu_l^{\otimes n}) = \operatorname{inj} \lim_{U \subset \overline{\{x\}}} H^p(U, \mu_l^{\otimes n})$$

where  $U$  runs through the open subsets of  $\overline{\{x\}}$ . Note that

$$\operatorname{inj} \lim_{U \subset \overline{\{x\}}} H^p(U, \mu_l^{\otimes n}) = H^p(\operatorname{Spec} k(x), \mu_l^{\otimes n})$$

To prove this we may assume that  $x$  is the generic point of  $X$  and that  $X$  is affine. We have  $\operatorname{Spec} k(x) = \operatorname{proj} \lim U$  where  $U$  runs through the open affines of  $X$ . We can now invoke [3, Thm VII.5.7] which says that under these circumstances etale cohomology commutes with inverse limits.

The main point of [5] is that the sheaffication for the Zariski topology of (5.1) degenerates at  $E_2$  and computes the sheaffication of the presheaf  $U \mapsto H^p(U, \mu_l^{\otimes n})$ . This will not concern us however.

If  $K$  is a field transcendence degree  $r$  over the algebraically closed field  $k$  then the  $l$ -cohomological dimension of  $K$  is  $r$ . See [18, p223 top]. The  $l$ -cohomological dimension is the degree above which all Galois cohomology of  $l$ -torsion sheaves vanishes. This means that  $E_1^{pq} = 0$  for  $q > \dim X$ .

We will now write out explicitly the coniveau spectral sequence in the case that  $X$  is a smooth proper connected surface over an algebraically closed field  $k$ .

$$\begin{array}{ccc} \bigoplus_{x \in X^{(0)}} H^2(x, \mu_l^{\otimes n}) & \bigoplus_{x \in X^{(1)}} H^1(x, \mu_l^{\otimes n-1}) & \bigoplus_{x \in X^{(2)}} H^0(x, \mu_l^{\otimes n-2}) \\ \bigoplus_{x \in X^{(0)}} H^1(x, \mu_l^{\otimes n}) & \bigoplus_{x \in X^{(1)}} H^0(x, \mu_l^{\otimes n-1}) & 0 \\ \bigoplus_{x \in X^{(0)}} H^0(x, \mu_l^{\otimes n}) & 0 & 0 \end{array}$$

So if  $C$  runs through the irreducible curves in  $X$  and  $x$  runs through the closed points of  $x$  we get a complex

$$(5.2) \quad H^2(X, \mu_l^{\otimes n}) \rightarrow H^2(k(X), \mu_l^{\otimes n}) \rightarrow \bigoplus_C H^1(k(C), \mu_l^{\otimes n-1}) \rightarrow \bigoplus_x \mu_l^{\otimes n-2} \rightarrow \mu_l^{\otimes n-2} \rightarrow 0$$

where we have used Poincare duality

$$H^4(X, \mu_l^{\otimes n}) \cong \mu_l^{\otimes n-2}$$

This complex is acyclic in positions 2,4,5 and has homology in  $H^3(X, \mu_l^{\otimes n})$  in position 3.

Since  $X$  is a smooth surface we have that  $\operatorname{Br}(X)_l \rightarrow \operatorname{Br}(k(X))_l$  is injective [18, Cor. IV.2.6]. From (4.1) applied to  $\operatorname{Spec} k(X)$  we get

$$H^2(k(X), \mu_l) = \operatorname{Br}(k(X))_l$$

Whence (5.2) transforms into a complex

$$(5.3) \quad 0 \rightarrow \mathrm{Br}(X)_l \rightarrow \mathrm{Br}(k(X))_l \xrightarrow{a} \bigoplus_C H^1(k(C), \mathbb{Z}/l\mathbb{Z}) \xrightarrow{r} \bigoplus_x \mu_l^{-1} \xrightarrow{s} \mu_l^{-1} \rightarrow 0$$

which is now also exact on the left.

In [4] an explicit description of the maps in this complex is given.<sup>3</sup> The map  $a$  is  $\bigoplus_C \mathrm{Ram}_C$  where

$$\mathrm{Ram}_C : \mathrm{Br}(k(X))_l \rightarrow H^1(k(C), \mathbb{Z}/l\mathbb{Z})$$

is the “standard ramification map” associated to the discrete valuation ring  $R \subset K \stackrel{\mathrm{def}}{=} \text{corresponding to } C \subset X$ . Note that we have

$$H^1(k(C), \mathbb{Z}/l\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(k(C)), \mathbb{Z}/l\mathbb{Z})$$

from which one deduces the well-known basic fact that the non-trivial elements of  $H^1(k(C), \mathbb{Z}/l\mathbb{Z})$  are represented by couples  $(L/k(C), \sigma)$  where  $L/k(C)$  is a cyclic extension of degree  $l$  and  $\sigma$  is a generator of  $\mathrm{Gal}(L/k(C))$  ( $\sigma$  is the inverse image of  $\bar{1} \in \mathbb{Z}/l\mathbb{Z}$ ). Hence if  $\mathrm{ram}_C$  takes a non-trivial value on  $[A] \in \mathrm{Br}(k(X))_l$  (i.e.  $A$  is “ramified” in  $C$ ) then  $\mathrm{ram}_C([A])$  defines a couple  $(L/k(C), \sigma)$  as above.

*Remark 5.2.* By a local computation one may show that the following ring theoretic description of the ramification map is correct [2]. Let  $D \in \mathrm{Br}(k(X))_l$  be a division algebra ramified in  $C$ . Let  $\Lambda \subset D$  be a maximal order over  $R$ . Let  $M$  be the twosided maximal ideal of  $\Lambda$ . Then  $\Lambda/M$  be is a central simple algebra. Let  $L$  be its center. One may show that  $M$  is generated by a normalizing element  $\Pi$ . Then  $\sigma = \Pi \cdot \Pi^{-1}$  defines an automorphism of  $L$  over  $k(C)$  and  $\mathrm{ram}_C([D]) = (L/k(C), \sigma)$ . See [21] for some of the unexplained terminology in this remark.

The map  $r$  is a direct sum  $\bigoplus_{C,x} r_{C,x}$  where  $x \in C$ . Assume that  $C$  is smooth and  $(L/k(C), \sigma) \in H^1(k(C), \mathbb{Z}/l\mathbb{Z})$ . Then  $r_{C,x}(L/k(C), \sigma)$  measures the ramification of  $L/k(C)$  in  $x$  in the sense that  $r_{C,x}(L/k(C), \sigma) = 0$  if and only if  $L/k(C)$  is unramified in  $x$  (in the classical sense of extensions of discrete valuation rings).

*Remark 5.3.* Using a local computation we can show that the following is correct. Let  $K = k(C)$  and let  $K_s$  be the separable closure of  $K$ . Then we have an exact sequence

$$0 \rightarrow \mu_l \rightarrow K_s^* \xrightarrow{l} K_s^* \rightarrow 0$$

from which we deduce using Hilbert’s theorem 90 (recall that over a field étale cohomology is the same as Galois cohomology).

$$H^1(K, \mu_l) = K^*/(K^*)^l$$

If  $v$  is a discrete valuation (associated to a point  $x \in C$ ) then we obtain an induced map

$$r : H^1(K, \mu_l) \rightarrow \mathbb{Z}/l\mathbb{Z} : \bar{s} \mapsto \overline{v(\bar{s})}$$

Since  $\mu_l \cong \mathbb{Z}/l\mathbb{Z}$  (non-canonically) we may twist  $r$  to obtain a (canonical) map

$$r' : H^1(K, \mathbb{Z}/l\mathbb{Z}) \rightarrow \mu_l$$

Then one has  $r' = r_{C,x}$ .

<sup>3</sup>The proof in [4] for the existence of this sequence is a bit different from, but equivalent to the one given here.

The map  $s$  is simply the sum map. Note that for arbitrary  $C$  we have a complex

$$H^1(k(C), \mathbb{Z}/l\mathbb{Z}) \rightarrow \bigoplus_{x \in C} \mu_l^{-1} \xrightarrow{s} \mu_l^{-1}$$

In other words is  $\xi \in H^1(k(C), \mathbb{Z}/l\mathbb{Z})$  then

$$\sum_{x \in C} r_{C,x}(\xi) = 0$$

This is a kind of reciprocity law.

The only reason why we need the coniveau complex is the following

**Proposition 5.4.** *Assume that  $0 \neq [A] \in \text{Br}(X)_l$  where  $X$  is a smooth connected surface over an algebraically closed field  $k$ . Then it is impossible for  $A$  to be ramified in a union of trees of  $\mathbb{P}^1$ 's.*

*Proof.* Assume that the ramification locus of  $A$  is a union of trees of  $\mathbb{P}^1$ 's. Then there is at least one of those  $\mathbb{P}^1$ 's which intersects the other components of the ramification locus in at most one point. Denote this  $\mathbb{P}^1$  by  $C$ . The case that there is no intersection point is very slightly easier so assume there is an intersection point and denote it by  $x$ . Finally put  $\xi = \text{Ram}_C(A)$ . If  $x \neq y \in C$  then there is no other component of the ramification locus which intersects  $y$  and hence it follows from (5.3) that  $r_{C,y}(\xi) = 0$ .

From the identity  $\sum_{x \in C} r_{C,x}(\xi) = 0$  we then obtain  $r_{C,x}(\xi) = 0$ . But this means that  $\xi$  is unramified over  $\mathbb{P}^1$ . To finish the proof we note that  $\mathbb{P}^1$  does not have any unramified coverings. See e.g. [18, I.5(f)].  $\square$

A particular type of surface singularity is a so-called rational double point. The minimal resolutions of such singularities are trees of  $\mathbb{P}^1$ 's whose incidence graph is a Dynkin diagram. See [17]. We obtain the following corollary to Proposition 5.4.

**Corollary 5.5.** *Assume that  $X$  is a proper, connected singular surface whose only singularities are rational double points. Let  $U$  be the complement of the singular locus and let  $\tilde{X} \rightarrow X$  be the minimal resolution of singularities. If  $\alpha \in \text{Br}(k(X))_l$  is unramified on  $U$  then  $\alpha$  is unramified on  $\tilde{X}$  as well.*

*Proof.* If  $\alpha$  is ramified on  $\tilde{X}$  then it can be at most be ramified on the exceptional locus which is a union of trees of  $\mathbb{P}^1$ 's. According to Proposition 5.4 it must then be unramified.  $\square$

The simplest rational double points are  $A_{l-1}$  singularities. They can be characterized as those singularities whose local ring  $(R, m)$  after completion becomes isomorphic to

$$k[[x, y, z]]/(z^l - xy)$$

( $l$  does not have to be prime the the characteristic here). An easy local computation shows that such an  $A_{l-1}$  singularity can be desingularized by  $\lceil (l-1)/2 \rceil$  consecutive blowups at singular points (which are themselves  $A_{l'-1}$  singularities for  $l' \leq l$ ). The resulting exceptional locus is the Dynkin diagram  $A_{l-1}$ .

**Example 5.6.** Let  $(R, m)$  be a regular local  $k$ -algebra of dimension two with residue field  $k$  and let  $f \in m^2$  be such that  $f - xy \cong 0 \pmod{m^3}$  where  $m = (x, y)$ . Then  $R = R[w]/(w^l - f)$  is an  $A_{l-1}$ -singularity. To see this note that after

completion we have  $R = k[[x, y]]$ . We claim that after redefining  $x, y$  we may assume  $f = xy$ . Assume

$$f - xy \in m^t$$

Put  $x' = x + p, y' = y + q$  such that  $p, q \in m^t$ . Then

$$f - x'y' \cong (f - xy) - xq - yp \pmod{m^{t+1}}$$

Since  $x, y$  generate  $m$  we may find  $p, q$  such that  $f - x'y' \in m^{t+1}$ . Repeating this procedure proves the claim.

Assume that  $X$  is a smooth proper connected surface and let  $\mathcal{L} \in \text{Pic}(X)$ . Let  $0 \neq s \in H^0(X, \mathcal{L}^l)$ . We define a sheaf of algebras on  $X$  as follows

$$\mathcal{A} = \bigoplus_{i=0}^{l-1} \mathcal{L}^{-i} T^i$$

where  $T^l = s$  and  $Y = \text{Spec} \mathcal{A}$ . Then  $Y \rightarrow X$  is a cyclic covering of degree  $l$  of  $X$  which is ramified in the zeroes of  $s$ .

**Proposition 5.7.** *Assume that the zero divisor  $H$  of  $s$  has normal crossings with smooth components. Let  $\tilde{Y}$  be the minimal resolution of singularities of  $Y$ . Let  $\alpha \in \text{Br}(k(X))_l$  and assume that the ramification locus of  $\alpha$  is contained in  $H$ . Then  $\alpha_{k(\tilde{Y})}$  is unramified.*

*Proof.* By Example 5.6  $Y$  has  $A_{l-1}$ -rational singularities. As the ramification locus of  $\alpha$  is contained in  $H$  it is standard that  $\alpha_{k(Y)}$  is unramified on the regular part of  $Y$  (e.g. [6, §23.3]). But then it follows from Corollary 5.5 that  $\alpha_{k(\tilde{Y})}$  is unramified as well.  $\square$

Before moving on we state a result which is peculiar to dimension two.

**Lemma 5.8.** *Let  $X$  be a smooth projective surface and let  $A$  be a central simple algebra over  $k(X)$  which is unramified. Then there exists an Azumaya algebra  $\mathcal{A}$  on  $X$  such that  $\mathcal{A}_{k(X)} = A$ .*

*Proof.* Let  $\mathcal{A}$  be a maximal order in  $A$ . I.e. a sheaf of  $\mathcal{O}_X$ -algebras torsion free and coherent as  $\mathcal{O}_X$ -module which is contained in  $A$  (such an object is called an “order”) and which is not properly contained in any other order. Maximal orders are a non-commutative version of integrally closed rings but they are not unique [21].

As an  $\mathcal{O}_X$  module we have

$$\mathcal{A} = \mathcal{A}^{\vee\vee} = \mathcal{H}om_X(\mathcal{H}om_X(\mathcal{A}, \mathcal{O}_X), \mathcal{O}_X)$$

since otherwise  $\mathcal{A}^{\vee\vee}$  would be a bigger order. A standard fact in commutative algebra says that a reflexive module over a ring of global dimension two is projective. So  $\mathcal{A}$  is locally free. Furthermore since  $A$  was supposed to be unramified we have that  $\mathcal{A}_x$  is Azumaya for all  $x \in X^{(1)}$  where  $X^{(1)}$  denotes the points of codimension one.

Consider now the standard map

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^\circ \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A})$$

This now an isomorphism in codimension one between locally free modules of the same rank. Hence it has to be an isomorphism.  $\square$

## 6. ARTIN SPLITTING

If  $\mathcal{A}$  is an Azumaya algebra of index  $d$  on a scheme  $X$ ,  $\mathcal{L}$  is an invertible sheaf on  $X$  and  $s \in H^0(X, \mathcal{A} \otimes_X \mathcal{L})$  then we denote the (reduced) characteristic polynomial of  $s$  by  $P_s(T)$ . The coefficient of  $T^i$  is a section of  $\mathcal{L}^{\otimes d-i}$ .

Put

$$L = \underline{\text{Spec}} \oplus_n \mathcal{L}^{-n} T^n$$

We may view  $P_s(t)\mathcal{L}^{-d}$  as sections of  $\oplus_n \mathcal{L}^{-n} T^n$ . Hence inside  $L$  they cut out a subscheme which we denote by  $Y_s$ . Looking e.g. locally it is easy to see that  $Y_s/X$  is finite and flat.

If we do not specify  $\mathcal{L}$  (e.g. in the local case) then we assume  $\mathcal{L} = \mathcal{O}_X$  (or rather that we have chosen some unspecified trivialization  $\mathcal{L} \cong \mathcal{O}_X$ ).

The following result is due to Mike Artin [2]

**Theorem 6.1.** *Assume that  $X$  is a smooth projective surface over an algebraically closed field  $k$ . Let  $\mathcal{A}$  be an Azumaya algebra of degree  $d$  over  $X$  which is generically a division algebra. Let  $\mathcal{M}$  be an ample line bundle on  $X$ . Then for  $r \gg 0$  and  $\mathcal{L} = \mathcal{M}^r$  there exists a Zariski open subset  $U \subset H^0(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L})$  such that for  $s \in U$  we have*

- (1)  $Y_s$  is a smooth connected surface.
- (2)  $Y_s/X$  is generically étale (i.e.  $k(Y_s)/k(X)$  is separable).
- (3)  $Y_s$  splits  $\mathcal{A}$ .

*Remark 6.2.* This can be extracted from [2, Thm 8.1.11]. However this last result is much more precise and it does not assume that  $\mathcal{A}$  is generically a division algebra.

The following key lemma analyzes the local situation.

**Lemma 6.3.** *Let  $(R, m)$  be a regular local  $k$ -algebra of dimension 2 with residue field  $k$ . Let  $A$  be an Azumaya algebra of degree  $d$  over  $R$ . Then there exists a closed subvariety  $W$  in  $A/m^2 A$  (the latter viewed as a vector space) of codimension 3 such that for  $s \in A$  we have that  $Y_s$  is regular if  $\bar{s} \notin W$  (here and below  $\bar{s}$  stands for  $s \bmod m^2$ ).*

*Proof.* We choose an isomorphism  $A/m^2 A \cong M_d(R/m^2)$  and an isomorphism  $R/m^2 \cong k[\eta, \zeta]/(\eta^2, \zeta^2, \eta\zeta)$ . Using these isomorphisms we let  $\text{GL}_d(k)$  act on  $A/m^2 A$  by conjugation.

We let  $W$  be the locus of elements  $\bar{s} = s_0 + s_1\eta + s_2\zeta$  in  $A/m^2 A$  such that  $s_0$  has a repeated eigenvalue  $\mu$  with the property  $\det(\bar{s} - \mu) = 0$ . We claim this is a closed subset of  $A/m^2 A$ . To prove this consider the subset  $\widetilde{W}$  of  $\mathbb{P}((A/m^2) \oplus k)$  defined by

$$\widetilde{W} = \{(\bar{s}, \mu) \mid \mu \text{ is a repeated eigenvalue of } s_0 \text{ and } \det(\bar{s} - \mu) = 0\}$$

This is closed since the condition that  $\mu$  is a repeated eigenvalue can be expressed as

$$\text{rk}(s_0 - \mu)^2 \leq d - 2$$

Hence  $\widetilde{W}$  is a projective scheme. The projection map  $\widetilde{W} \rightarrow \mathbb{P}(A/m^2 A) : (\bar{s}, \mu) \mapsto \bar{s}$  is well defined since if  $\bar{s} = 0$  then  $\det(\mu) = 0$  and hence  $\mu = 0$ . Hence the image of  $\widetilde{W}$  in  $\mathbb{P}(A/m^2 A)$  is closed. Since  $W$  is the cone of the image of  $\widetilde{W}$  in  $\mathbb{P}(A/m^2 A)$  it follows that  $W$  is also closed.

We can describe the elements of  $W$  by putting  $s_0$  in Jordan normal form. One verifies that  $\bar{s} \in W$  if and only if one of the following conditions holds

- (1)  $s_0$  has two Jordan blocks with equal eigenvalues.
- (2)  $s_0$  has a Jordan block of size  $> 1$  of the form

$$\begin{pmatrix} \mu & 1 & \cdots & 0 & 0 \\ 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \mu & 1 \\ * & 0 & \cdots & 0 & \mu \end{pmatrix}$$

such that  $\bar{s}_{1,ij} = \bar{s}_{2,ij} = 0$  where  $(i,j)$  refers to the entry of  $\bar{s}$  marked by “\*” in the above matrix.

From this description it follows easily that  $W$  has codimension three.

Assume now  $s \in A$  such that  $\bar{s} \notin W$ . Denote the closed point of  $\text{Spec } R$  by  $o$ .

We will describe the equation of  $Y_s$  in the neighborhood of a closed point  $(o, \mu) \in Y_s$  where  $\mu$  is an eigenvalue of  $s_0$ . Replacing  $s$  by  $s - \mu$  we may assume  $\mu = 0$ . The equation of  $Y_s$  around  $(o, 0)$  will be

$$P_s(T) = T^d + a_1 T^{d-1} + \cdots + a_{d-1} T + a_d$$

where  $a_i \in R$ . We have

$$\begin{aligned} a_d &\cong \pm \det(\bar{s}) \pmod{m^2} \\ a_{d-1} &\cong \pm \text{tr}(\text{ad}(\bar{s})) \pmod{m^2} \end{aligned}$$

To be able to compute we put  $s_0$  in Jordan normal form. We separate two cases.

- (1) The eigenvalue 0 has multiplicity one. In that case  $a_{d-1} \in R^*$ .
- (2) The eigenvalue 0 has higher multiplicity. Since  $\bar{s} \notin W$  we obtain from the definition of  $W$ :  $\bar{a}_d = \det(\bar{s}) \neq 0$ . In particular  $a_d \notin m^2$ .

Hence in both cases  $P_s(T) \notin (m, T)^2 \subset R[T]_{(T)}$ . Thus  $Y_s$  is regular in  $(o, 0)$ .  $\square$

We need a trivial lemma about ample linebundles.

**Lemma 6.4.** *Let  $\mathcal{M}$  be ample on  $X$  and let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Then for  $r \gg 0$  and for  $\mathcal{L} = \mathcal{M}^r$  we have that for all  $x$  the map*

$$H^0(X, \mathcal{F} \otimes_X \mathcal{L}) \rightarrow H^0(X, \mathcal{F}/m_x^2 \mathcal{A} \otimes_X \mathcal{L})$$

is surjective.

*Proof.* For  $s \gg 0$  we have that  $\mathcal{M}^s$  is very ample [15], i.e. for all  $x \in X$  we have that

$$H^0(X, \mathcal{M}^s) \rightarrow H^0(X, \mathcal{M}^s \otimes \mathcal{O}_X/m_x^2)$$

is surjective. See [15, Prop 7.3].

For  $t \gg 0$  we have that  $\mathcal{F} \otimes \mathcal{M}^t$  is generated by global sections [15]. I.e. there is a surjective map

$$\mathcal{O}_X^{\oplus N} \rightarrow \mathcal{F} \otimes \mathcal{M}^t$$

and hence surjective maps

$$\begin{aligned} (\mathcal{M}_X^s)^{\oplus N} &\rightarrow \mathcal{F} \otimes \mathcal{M}^{s+t} \\ (\mathcal{M}_X^s \otimes \mathcal{O}_X/m_x^2)^{\oplus N} &\rightarrow \mathcal{F} \otimes \mathcal{M}^{s+t} \otimes \mathcal{O}_X/m_x^2 \end{aligned}$$

Since the last map is between sheaves concentrated in a point, it remains surjective after applying  $H^0(X, -)$ .

The lemma now follows from the following commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{M}^s)^{\oplus N} & \longrightarrow & H^0(X, \mathcal{F} \otimes \mathcal{M}^{s+t}) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{M}^s \otimes \mathcal{O}_X/m_x^2)^{\oplus N} & \longrightarrow & H^0(X, \mathcal{F} \otimes \mathcal{M}^{s+t} \otimes \mathcal{O}_X/m_x^2) \end{array}$$

using the fact that the lower horizontal and the leftmost vertical maps are surjective.  $\square$

*Proof of Theorem 6.1.* We prove smoothness of  $Y_s$  first. By Lemma 6.4 we have that for  $r \gg 0$  the map

$$p : H^0(X, \mathcal{A} \otimes_X \mathcal{L}) \rightarrow H^0(X, \mathcal{A}/m_x^2 \mathcal{A} \otimes_X \mathcal{L})$$

is surjective for all  $x$ . Let  $W_x \subset H^0(X, \mathcal{A}/m_x^2 \mathcal{A} \otimes_X \mathcal{L})$  be the subvariety we denoted by  $W$  in Lemma 6.3 (choosing an arbitrary trivialization of  $\mathcal{L}$  in a neighborhood of  $x$ ).

Let  $s_1, \dots, s_n$  be a basis for  $H^0(X, \mathcal{A} \otimes_X \mathcal{L})$ . Put  $\mathbb{A}^n = \text{Spec } k[t_1, \dots, t_n]$  and put  $\tilde{s} = t_1 s_1 + \dots + t_n s_n$ . Let  $\psi : Y_{\tilde{s}} \rightarrow X \times \mathbb{A}^n$  be the ramified cover defined by  $\tilde{s}$ . Let  $\text{NSm}(Y_{\tilde{s}})$  be the locus where the projection  $Y_{\tilde{s}} \rightarrow \mathbb{A}^n$  is not smooth (i.e. where  $\Omega_{Y_{\tilde{s}}/\mathbb{A}^n}$  is not locally free). We need to prove that the image of  $\text{NSm}(Y_{\tilde{s}})$  is not  $\mathbb{A}^n$ .

For  $x \in X$  we have shown above that  $\psi^{-1}(x \times \mathbb{A}^n) \cap \text{NSm}(Y_{\tilde{s}}) \subset \psi^{-1}(x \times p^{-1}(W_x))$ . It follows that  $\text{NSm}(Y_{\tilde{s}})$  has codimension  $\geq 3$  in  $Y_{\tilde{s}}$ . Since  $\dim X = 2$  this means that the image of  $\text{NSm}(Y_{\tilde{s}})$  cannot be the full  $\mathbb{A}^n$ . This finishes the proof of smoothness.

Now we prove splitting and generic etaleness. Fix  $x \in X$ . There is a Zariski open  $\tilde{V} \subset A/mA \cong M_d(k)$  such that for  $t \in \tilde{V}$  we have that the characteristic polynomial of  $t$  has  $d$  distinct roots. Let  $V$  be the inverse image of  $\tilde{V}$  under the surjective map

$$p' : H^0(X, \mathcal{A} \otimes_X \mathcal{L}) \rightarrow H^0(X, \mathcal{A}/m_x \mathcal{A} \otimes_X \mathcal{L})$$

Hence  $Y_s/X$  is unramified in  $x$  if  $s \in V$ . Since  $Y_s/X$  is flat we deduce that  $Y_s/X$  is etale in a neighborhood of  $x$ . Hence  $Y_s/X$  is generically etale if  $s \in V$ .

If  $k(Y_s)$  does not split  $D = \mathcal{A}_\eta$  then the map

$$k(Y_s) \rightarrow D : T \mapsto s$$

must land in a non-maximal subfield of  $D$ . But then  $P_s(T)$  is not the minimal polynomial of  $s$  and hence it has multiple roots. It follows that  $Y_s/X$  is not generically etale.  $\square$

## 7. ELEMENTARY TRANSFORMATIONS

Suppose  $X$  is a scheme and  $I$  is an invertible ideal in  $\mathcal{O}_X$ . Let  $\mathcal{A}$  be an Azumaya algebra on  $X$  and assume that  $\bar{\mathcal{A}} = \mathcal{A}/I\mathcal{A} = \text{End}_{\mathcal{O}_D}(\bar{V})$  with  $\bar{V}$  a vector bundle on  $D = V(I)$ . Suppose we have a subbundle  $\bar{F}$  of  $\bar{V}$ .

We define

$$\bar{\mathcal{B}} = \{\phi \in \text{End}_{\mathcal{O}_D}(\bar{V}) \mid \phi(\bar{F}) \subset \bar{F}\}$$

and we let  $\mathcal{B}$  be the inverse image of  $\bar{\mathcal{B}}$  in  $\mathcal{A}$ . Define

$$\bar{J} = \{\phi \in \text{End}_{\mathcal{O}_D}(\bar{V}) \mid \phi(\bar{V}) \subset \bar{F}\}$$

and let  $J$  be the inverse image of  $\bar{J}$  in  $\mathcal{A}$ . Thus we have inclusions

$$I\mathcal{A} \subset J \subset \mathcal{B} \subset \mathcal{A}$$

It is clear that  $J$  is a twosided ideal in  $\mathcal{B}$ . It is also a right  $\mathcal{A}$  ideal. We put

$$\mathcal{A}' = \text{End}_{\mathcal{A}}(J)$$

and call  $\mathcal{A}'$  the *elementary transform* of  $\mathcal{A}$  with respect to the data  $(D, \mathcal{F}, \mathcal{V})$ .

*Remark 7.1.* Max Lieblich points out that this definition can be generalized and in this way actually becomes more transparent. Instead of giving  $\mathcal{V}$  and  $\mathcal{F}$  as input one may start directly with a locally free right ideal  $\bar{J} \in \bar{\mathcal{A}}$  which is locally a direct summand. In this way we do not have to assume that  $\bar{\mathcal{A}}$  is split. In these notes we stick to the original definition.

To analyze the properties of elementary transformations we may work locally for the etale topology. Switching to an affine setting with  $X = \text{Spec } R$  we may assume that  $\mathcal{A} = M_n(R)$ ,  $\bar{V} = \bar{R}^n$ ,  $\bar{F} = \bar{R}^k$ . Then we find

$$J = \begin{pmatrix} I & \cdots & I & I & \cdots & I \\ \vdots & & \vdots & \vdots & & \vdots \\ I & \cdots & I & I & \cdots & I \\ R & \cdots & R & R & \cdots & R \\ \vdots & & \vdots & \vdots & & \vdots \\ R & \cdots & R & R & \cdots & R \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} R & \cdots & R & I & \cdots & I \\ \vdots & & \vdots & \vdots & & \vdots \\ R & \cdots & R & I & \cdots & I \\ R & \cdots & R & R & \cdots & R \\ \vdots & & \vdots & \vdots & & \vdots \\ R & \cdots & R & R & \cdots & R \end{pmatrix}$$

$$\mathcal{A}' = \begin{pmatrix} R & \cdots & R & I & \cdots & I \\ \vdots & & \vdots & \vdots & & \vdots \\ R & \cdots & R & I & \cdots & I \\ I^{-1} & \cdots & I^{-1} & R & \cdots & R \\ \vdots & & \vdots & \vdots & & \vdots \\ I^{-1} & \cdots & I^{-1} & R & \cdots & R \end{pmatrix}$$

We deduce immediately

**Lemma 7.2.** *The elementary transform of an Azumaya algebra is an Azumaya algebra.*

Now we work globally again. Let  $\bar{Q} = \bar{V}/\bar{F}$ . Directly from the definition we deduce that there is a short exact sequence

$$0 \rightarrow \mathcal{B} \rightarrow \mathcal{A} \rightarrow \text{Hom}_{\bar{R}}(\bar{F}, \bar{Q}) \rightarrow 0$$

We will now construct a similar exact sequence for  $\mathcal{A}'$ . The inclusion  $I\mathcal{A} \subset J$  yields a map

$$\mathcal{A}' = \text{Hom}_{\mathcal{A}}(J, J) \rightarrow I^{-1}J = I^{-1} \otimes J \rightarrow I^{-1} \otimes_{\mathcal{O}_D} \bar{J}$$

One checks locally that the image of this map is precisely

$$I^{-1} \otimes_{\mathcal{O}_D} \mathrm{Hom}_{\mathcal{O}_D}(\bar{Q}, \bar{F})$$

and likewise one checks locally that the kernel of the induced map

$$\mathcal{A}' \rightarrow I^{-1} \otimes_{\mathcal{O}_D} \mathrm{Hom}_{\mathcal{O}_D}(\bar{Q}, \bar{F})$$

is precisely  $\mathcal{B}$ . Taking into account that

$$I^{-1} \otimes_{\mathcal{O}_D} \mathrm{Hom}_{\mathcal{O}_D}(\bar{Q}, \bar{F}) = \mathrm{Hom}_{\mathcal{O}_D}(I/I^2 \otimes_{\mathcal{O}_D} \bar{Q}, \bar{F})$$

we get an exact sequence

$$0 \rightarrow \mathcal{B} \rightarrow \mathcal{A}' \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathrm{scr}_D}}(I/I^2 \otimes_{\mathcal{O}_D} \bar{Q}, \bar{F}) \rightarrow 0$$

There is a proof of the following proposition in [2]. We give the proof in [6].

**Proposition 7.3.** *Let  $X$  be a scheme,  $\mathcal{A} \in \mathrm{Az}_n(X)$ , with  $n$  invertible on  $X$  and  $I \subset \mathcal{O}_X$  an invertible ideal. Assume there is a  $\mathcal{O}_D = \mathcal{O}_X/I$  vector bundle  $\bar{V}$  such that  $\bar{\mathcal{A}} = \mathcal{A}/IA = \mathcal{E}nd(\bar{V})$ . Let  $\bar{F} \subset \bar{V}$  be a subbundle of constant rank  $r$ . Let  $\mathcal{A}'$  be the associated elementary transform. Then*

$$(7.1) \quad \mathrm{cl}(\mathcal{A}') = \mathrm{cl}(\mathcal{A}) + r[I] \in H^2(X, \mu_n)$$

where  $[I]$  is the image of  $I$  under the composition

$$\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(X)/n \rightarrow H^2(X, \mu_n)$$

*Proof.* We first consider the case that  $\mathcal{A} = \mathrm{End}_X(V)$  is split,  $\bar{V} = V/IV$  and  $\bar{F} \subset \bar{V}$ . Let  $\bar{Q} = \bar{V}/\bar{F}$  and let  $W$  be the kernel of  $V \rightarrow \bar{Q}$ . Then  $J = \mathrm{Hom}_X(V, W)$  and hence  $\mathcal{A}' = \mathrm{End}_X(W)$ .

Hence according to Lemma 4.2  $\mathrm{cl}(\mathcal{A}) = -[\wedge^n V]$ ,  $\mathrm{cl}(\mathcal{A}') = -[\wedge^n W]$ . So we have to compare  $\wedge^n V$  and  $\wedge^n W$ . The inclusion  $W \subset V$  certainly yields an inclusion  $i : \wedge^n W \subset \wedge^n V$ .

Working locally we may assume  $V = \mathcal{O}_X^n$  and  $W = I^{n-r} \oplus \mathcal{O}_X^r$ . We then find that the image of  $i$  is equal to  $I^{n-r}(\wedge^n V)$ . Hence

$$[\wedge^n W] = (n-r)[I] + [\wedge^n V] \cong -r[I] + [\wedge^n V]$$

(we are working modulo  $n$ ). Hence by lemma 4.2

$$\mathrm{cl}(\mathcal{A}') = r[I] + \mathrm{cl}(\mathcal{A})$$

Before we continue we mention that we could have taken  $\bar{V} = J \otimes V/IV$ ,  $J \in \mathrm{Pic}(D)$  and  $\bar{F} \subset J \otimes V/IV$ , but then the elementary transform with respect to  $\bar{F}$  is the same as the elementary transform with respect to  $J^{-1} \otimes \bar{F} \subset V/IV$  and  $\mathrm{rk}_{\bar{R}}(\bar{F}) = \mathrm{rk}_{\bar{R}}(J^{-1} \otimes \bar{F})$ . So our assumption that  $\bar{V} = V/IV$  was not a restriction.

Now we assume that  $\mathcal{A}$  is general. The next argument is due to Gille. Let  $\pi : Y \rightarrow X$  be the Brauer-Severi associated to  $\mathcal{A}$ . Thus  $Y$  splits  $\mathcal{A}$  and etale locally  $Y$  is a  $\mathbb{P}^{n-1}$  bundle over  $X$ .

It is easy to see that elementary transform is compatible with base change. Thus  $\pi^*(\mathcal{A}')$  is the elementary transform of  $\pi^*\mathcal{A}$  with respect to  $\pi^*\bar{F}$ . Since  $\pi^*(\mathcal{A})$  is split we find

$$\pi^* \mathrm{cl}(\mathcal{A}') = \mathrm{cl}(\pi^*(\mathcal{A}')) = r[\pi^*I] + \mathrm{cl}(\pi^*(\mathcal{A})) = r\pi^*[I] + \pi^*(\mathrm{cl}(\mathcal{A}))$$

Hence it sufficient to show that

$$\pi^* : H^2(X, \mu_n) \rightarrow H^2(Y, \mu_n)$$

is injective. We compute  $H^2(Y, \mu_n)$  using the Leray spectral sequence

$$H^p(X, R^q \pi_* \mu_n) \Rightarrow H^{p+q}(Y, \mu_n)$$

We have  $R^0 \pi_* \mu_n = \mu_n$ ,  $R^1 \pi_* \mu_n = 0$  (see below). Hence the  $E_2$ -page of the spectral sequence is as follows

$$\begin{array}{ccccc} H^0(X, R^2 \pi_* \mu_n) & H^1(X, R^2 \pi_* \mu_n) & H^2(X, R^2 \pi_* \mu_n) & & \\ 0 & 0 & 0 & & \\ H^0(X, \mu_n) & H^1(X, \mu_n) & H^2(X, \mu_n) & & \end{array}$$

so that we have an exact sequence

$$0 \rightarrow H^2(X, \mu_n) \rightarrow H^2(Y, \mu_n) \rightarrow H^0(X, R^2 \pi_* \mu_n)$$

and in particular we get the requested injectivity.  $\square$

**Lemma 7.4.** *Let  $Y \rightarrow X$  be a relative Brauer-Severi scheme. Then we have*

$$\begin{aligned} R^0 \pi_* \mu_n &= \mu_n \\ R^1 \pi_* \mu_n &= 0 \end{aligned}$$

*Proof.* Using the proper base change theorem [11, IV-1] it suffices to prove this in case  $X$  is a geometric point. But then  $Y = \mathbb{P}^{n-1}$  and the cohomology of projective space is well-known (e.g [18, Example VI.5.6]).  $\square$

*Remark 7.5.* In [6] the formula (7.1) has a  $-$  sign. This is because it is assumed  $I = \mathcal{O}_X(-D)$  for a Cartier divisor  $D$  and the formula is in terms of  $D$ .

*Remark 7.6.* Artin and de Jong [2] show that two Azumaya algebras on a surface which are birational can be transformed into each other by an elementary transform based on a smooth curve.

## 8. KILLING OBSTRUCTIONS

The following we take from de Jong [8, lemma 3.1].

**Lemma 8.1.** *Let  $X \rightarrow X'$  be a closed immersion defined by an ideal  $I$  of square zero. Let  $\mathcal{A}$  be an Azumaya algebra on  $X$ . If  $H^2(X, I \otimes_X (\mathcal{A}/\mathcal{O}_X)) = 0$  then  $\mathcal{A}$  lifts to  $X'$ .*

*Proof.* Etale locally  $\mathcal{A}$  is a matrix algebra. A matrix algebra can obviously be lifted. Furthermore any such lifting is itself a matrix algebra (since we may lift idempotents).

It follows that etale locally  $\mathcal{A}$  lifts uniquely up to isomorphism. The sheaf of  $\mathcal{O}_{X'}$  isomorphisms of a given lift reducing to the identity on  $\mathcal{A}$  is  $I \otimes_X \text{Der}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{A})$ . It is part of the standard formalism of deformation theory that the obstructions to gluing local lifts are lying in the  $H^2$  of this sheaf.

We finish by observing that there is an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{A} \rightarrow \text{Der}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{A}) \rightarrow 0$$

This is basically Skolem-Noether for derivations.  $\square$

*Remark 8.2.* Etale cohomology and Zariski cohomology coincide for quasi-coherent sheaves [18, Remark 3.8]. So for computing the cohomology group  $H^2(X, I \otimes_X (\mathcal{A}/\mathcal{O}_X))$  we may use Zariski cohomology if we want to.

**Lemma 8.3.** *Let  $D/k$  be a smooth connected projective curve and  $\mathcal{V}$  a vector bundle of rank  $n \geq 2$  on  $D$ . Let  $T \subset D(k)$  be finite and assume given for every  $t \in T$  a one dimensional subspace  $F_t \subset V_t$ . Then there exists a sub line bundle  $\mathcal{F} \subset \mathcal{V}$  such that for  $t \in T$  we have  $\mathcal{F}_t = F_t$ . Moreover, fixing the other data,  $\mathcal{F}$  can be chosen to be of arbitrarily negative degree.*

*Proof.* Define  $\mathcal{W}$  by

$$0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \rightarrow \bigoplus_{t \in T} i_{t,*}(V_t/F_t) \rightarrow 0$$

Choose a line bundle  $\mathcal{F}$  such that  $\mathcal{W} \otimes \mathcal{F}^{-1}$  is generated by global sections. Let  $V = \text{Hom}_X(\mathcal{F}, \mathcal{W})$ . We have that the following map is surjective for all  $x \in X(k)$

$$V = H^0(X, \mathcal{W} \otimes_X \mathcal{F}^{-1}) \rightarrow \mathcal{W}_x \otimes_k \mathcal{F}_x = \text{Hom}_k(\mathcal{F}_x, \mathcal{W}_x)$$

In particular  $\dim V \geq n$ .

An element  $s \in V$  defines a map  $\mathcal{F} \rightarrow \mathcal{W} \rightarrow \mathcal{V}$  as in the statement of the lemma if the following two conditions hold

- (1) If  $x \notin T$  then  $s_x$  is not zero.
- (2) If  $x \in T$  then the composition  $\mathcal{F}_x \xrightarrow{s_x} \mathcal{W}_x \rightarrow F_x$  is not zero.

Let  $d = \dim V$ . Condition (2) is true on a Zariski open  $U$  of  $V$  ( $U$  is the complement of a finite number of hyperplanes). Let  $Z \subset (D - T) \times U$  be the set of pairs  $(x, s)$  such that  $s_x = 0$ . Clearly  $Z$  is closed. The dimension of the fibers  $Z \rightarrow D - T$  is  $d - n$ . Thus  $\dim Z \leq 1 + d - n$ . Since  $n \geq 2$  this is less than  $\dim U = d$ . It now suffices to select an  $s$  not in the image of the projection  $Z \rightarrow U$ .  $\square$

**Theorem 8.4.** *Let  $X/k$  be a smooth connected proper surface. Let  $\mathcal{A} \in \text{Az}_n(X)$ . Then there is an elementary transformation  $\mathcal{A}' \in \text{Az}_n(X)$  of  $\mathcal{A}$  such that*

$$\text{cl}(\mathcal{A}) = \text{cl}(\mathcal{A}')$$

and

$$H^2(X, \mathcal{A}'/\mathcal{O}_X) = 0$$

*Proof.* Using the reduced trace map we see that  $\mathcal{A}'/\mathcal{O}_X$  is self dual. Hence it is sufficient to construct  $\mathcal{A}'$  in such a way that

$$H^0(X, \mathcal{A}'/\mathcal{O}_X \otimes_X \omega_X) = 0$$

For an arbitrary line bundle  $\mathcal{L}$  on  $X$  we will construct  $\mathcal{A}'$  in such a way that

$$H^0(X, \mathcal{A}'/\mathcal{O}_X \otimes_X \mathcal{L}) = 0$$

or equivalently (since  $\mathcal{O}_X$  is a factor of  $\mathcal{A}$ ):

$$H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{A}' \otimes_X \mathcal{L})$$

is an isomorphism.

Assume  $s \in H^0(X, \mathcal{A} \otimes_X \mathcal{L}) \setminus H^0(X, \mathcal{L})$ . There exists  $t \in X$  such that  $s_t \notin \mathcal{L}_t$ . We have  $\mathcal{A}_t = \text{End}_k(V_t \otimes \mathcal{L}_t)$  for  $V_t$  a  $n$ -dimensional vector space. Hence there is a one dimensional subspace  $F_t \subset V_t$  such that  $s_t(F_t) \not\subset F_t \otimes \mathcal{L}_t$ . If  $s' \in H^0(X, \mathcal{A} \otimes_X \mathcal{L}) \setminus H^0(X, \mathcal{L})$  is such that  $s'_t(F_t) \subset F_t \otimes \mathcal{L}_t$  then we can find  $t' \in X$ ,  $F_{t'} \subset V_{t'}$  such that  $s'_{t'}(F_{t'}) \not\subset F_{t'} \otimes \mathcal{L}_{t'}$ .

Continuing we find a finite subset  $T \subset X$  and one-dimensional subspaces  $(F_t \subset V_t)_{t \in T}$

$$H^0(X, \mathcal{L}) = \{s \in H^0(X, \mathcal{A} \otimes_X \mathcal{L}) \mid \forall t \in T : s_t(F_t) \not\subset F_t \otimes \mathcal{L}_t\}$$

Choose an ample line bundle  $\mathcal{L}$  on  $X$ . For  $q$  sufficiently big a generic section of  $\mathcal{L}^{\otimes nq}$  vanishing on  $T$  will define a smooth curve  $D \subset X$  passing through  $T$  (this is a version of Bertini, see below). Thus we have  $\mathcal{L}^{\otimes nq} \cong \mathcal{O}_X(D)$ . In particular the class of  $D$  in  $\text{Pic}(X)$  is divisible by  $n$ .

By Tsen's theorem there is a vector bundle  $\bar{V}$  of rank  $n$  on  $D$  such that  $\mathcal{A}_D = \mathcal{E}nd_D(\bar{V})$ . For  $t \in T$  we have now two isomorphisms  $\mathcal{A}_t \cong \text{End}(V_t)$ ,  $\mathcal{A}_t \cong \text{End}(\bar{V}_t)$ . These two isomorphisms are related to each other through an isomorphism  $\eta_t : V_t \cong \bar{V}_t$ .

According to Lemma 8.3 we may choose a sub line bundle  $\bar{\mathcal{F}} \subset \bar{V}$  such that for all  $t \in T$  we have  $\bar{\mathcal{F}}_t = \eta_t(F_t)$ . Let  $\mathcal{A}'$  be the elementary transform of  $\mathcal{A}$  associated to  $(D, \bar{\mathcal{F}}, \bar{V})$ .

Since  $D$  is in  $n \text{Pic}(X)$  we have  $\text{cl}(\mathcal{A}) = \text{cl}(\mathcal{A}')$  (using (7.1)). Let  $\bar{\mathcal{Q}} = \bar{V}/\bar{\mathcal{F}}$ .

We have an exact sequence

$$0 \rightarrow \mathcal{B} \otimes_X \mathcal{L} \rightarrow \mathcal{A} \otimes_X \mathcal{L} \rightarrow \mathcal{H}om_D(\bar{\mathcal{F}}, \bar{\mathcal{Q}}) \otimes \mathcal{L}_D \rightarrow 0$$

Checking at the points of  $T$  we see that

$$H^0(X, \mathcal{B} \otimes_X \mathcal{L}) = H^0(X, \mathcal{L})$$

Now we consider the other sequence

$$0 \rightarrow \mathcal{B} \otimes_X \mathcal{L} \rightarrow \mathcal{A}' \otimes_X \mathcal{L} \rightarrow \mathcal{H}om_D(I_D/I_D^2 \otimes \bar{\mathcal{Q}}, \bar{\mathcal{F}}) \otimes_D \mathcal{L}_D \rightarrow 0$$

where  $I_D = \mathcal{O}(-D)$ . We deduce that there is an exact sequence

$$0 \rightarrow H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{A}' \otimes_X \mathcal{L}) \rightarrow \text{Hom}_D(I_D/I_D^2 \otimes_D \bar{\mathcal{Q}}, \bar{\mathcal{F}}) \otimes_D \mathcal{L}_D$$

Now

$$\text{Hom}_D(I_D/I_D^2 \otimes_D \bar{\mathcal{Q}}, \bar{\mathcal{F}} \otimes_X \mathcal{L}) \subset \text{Hom}_D(I_D/I_D^2 \otimes_D \bar{V}, \bar{\mathcal{F}} \otimes_X \mathcal{L}_D)$$

Choosing  $\bar{\mathcal{F}}$  negative enough we get

$$\text{Hom}_D(I_D/I_D^2 \otimes_D \bar{V}, \bar{\mathcal{F}} \otimes_X \mathcal{L}_D) = 0$$

This finishes the proof.  $\square$

We have used the following version Bertini.

**Lemma 8.5.** *Let  $X/k$  be a smooth projective variety of  $\dim X \geq 2$ . Let  $\mathcal{L}$  be an ample line bundle on  $X$ . Let  $T$  be a finite subset of  $X$ . Then for  $n$  large enough the zeroes of a generic section of  $\mathcal{L}^{\otimes n}$  which is zero on  $T$  will be smooth and connected.*

*Proof.* Connectedness follows from [15, Thm III.7.9]. We prove smoothness by suitably adapting [15, Thm II.8.18].

If  $\mathcal{M}$  is a very ample line bundle on  $X$  then for all  $x \in X$  we have that

$$H^0(X, \mathcal{M}) \rightarrow H^0(X, \mathcal{M} \otimes \mathcal{O}_X/m_x^2)$$

is surjective. This is in fact an equivalence. See [15, Prop 7.3]. It follows easily that if  $\mathcal{E}$  is a coherent sheaf generated by global sections then

$$H^0(X, \mathcal{E} \otimes_X \mathcal{M}) \rightarrow H^0(X, \mathcal{E} \otimes \mathcal{M} \otimes \mathcal{O}_X/m_x^2)$$

is surjective as well.

It follows from Lemma 6.4 that for  $n \gg 0$  we have that

$$H^0(X, \mathcal{I} \otimes_X \mathcal{L}^n) \rightarrow H^0(X, \mathcal{I} \otimes \mathcal{L}^n \otimes \mathcal{O}_X/m_x^2) = H^0(X, (\mathcal{I} \otimes \mathcal{L}^n)/(\mathcal{I} \otimes \mathcal{L}^n)_m_x^2)$$

is surjective for all  $x$ . Imitating the proof of [15, Thm II.8.18] we find that there is a Zariski dense subset  $U \subset H^0(X, \mathcal{I} \otimes_X \mathcal{L}^n)$  such that for  $s \in U$  and  $x \notin T$  we

have that the zeroes  $Z$  of  $s$  are smooth in  $x$ . Naturally  $Z$  passes through  $T$ . It remains to prove that can make  $Z$  smooth in  $T$  as well. Let  $t \in T$ . Then  $Z$  is not smooth in  $t$  if  $s_t \in H^0(X, \mathcal{L}^n \otimes m_t^2) = H^0(X, (\mathcal{I} \otimes \mathcal{L}^n)m_t)$ . Hence by taking  $s$  in the complement of a suitable set of linear spaces  $Z$  is smooth in  $t$  as well. We are done.  $\square$

## 9. GENERAL LIFTING

**Proposition 9.1.** *Assume that we have a proper flat map of finite type  $\phi : W \rightarrow C$  with  $C/k$  of finite type as well. Let  $x \in C$  and  $X = \phi^{-1}(x)$ . Let  $\mathcal{A}_0 \in \text{Az}_n(X)$  be such that*

$$H^2(X, \mathcal{A}_0/\mathcal{O}_X) = 0$$

Then there exists a diagram

$$\begin{array}{ccccc} X = \phi'^{-1}(x) = \phi^{-1}(x) & \longrightarrow & W' = C' \times_C W & \longrightarrow & W \\ \downarrow & & \phi' \downarrow & & \downarrow \phi \\ x & \longrightarrow & C' & \xrightarrow{\text{etale}} & C \end{array}$$

together with  $\mathcal{A}' \in \text{Az}_n(W')$  such that  $\mathcal{A}'_{\phi'^{-1}(x)} = \mathcal{A}_0$ .

*Proof.* Let  $m_x \subset \mathcal{O}_C$  be the defining ideal for  $x \in C$ . The defining ideal of  $X_0 \stackrel{\text{def}}{=} X \subset W$  is

$$I \stackrel{\text{def}}{=} m_x \mathcal{O}_W \stackrel{\text{flatness}}{\cong} m_x \otimes_{\mathcal{O}_C} \mathcal{O}_W$$

Let  $X_n$  be defined by  $I^{n-1} \cong m_x^{n-1} \otimes_C \mathcal{O}_W$ . Then  $X_{n+1}$  inside  $X_n$  is defined by

$$I^{n-1}/I^n \cong m_x^{n-1}/m_x^n \otimes_C \mathcal{O}_W \cong (\mathcal{O}_x)^{\oplus m_n} \otimes_C \mathcal{O}_W \cong \mathcal{O}_{X_0}^{\oplus m_n}$$

Hence the obstruction of lifting an  $\mathcal{A}_n \in \text{Az}(X_n)$  to an  $\mathcal{A}_{n+1} \in \text{Az}(X_{n+1})$  lies in

$$H^2(X_0, \mathcal{O}_{X_0}^{\oplus m_n} \otimes_{X_0} \mathcal{A}_0/\mathcal{O}_{X_0}) = 0$$

We may construct liftings

$$\cdots \rightarrow \mathcal{A}_m \rightarrow \cdots \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}_0$$

where  $\mathcal{A}_m$  lives in  $\text{Az}_n(X_m)$ . Put

$$\hat{\mathcal{A}} = \text{proj lim}_n \mathcal{A}$$

$\hat{\mathcal{A}}$  is an Azumaya algebra over the formal completion  $\hat{X}$  of  $W$  at  $X$ . Recall that the formal completion of  $X$  in  $W$  is the ringed space  $(X, \hat{\mathcal{O}}_{W,X})$ . This is not a scheme but a so-called *formal scheme* [15].

By Grothendieck's existence theorem [12, 5.1.4] there is an equivalence  $\text{coh}(\text{Spec } \hat{\mathcal{O}}_{C,x} \times_C W) \cong \text{coh}(\hat{X})$ . Note that  $\text{Spec } \hat{\mathcal{O}}_{C,x} \times_C W$  is an actual scheme.

From the Grothendieck existence theorem we deduce easily  $\text{Az}_n(\text{Spec } \hat{\mathcal{O}}_{C,x} \times_C W) = \text{Az}_n(\hat{X})$ . Let  $\tilde{\mathcal{A}} \in \text{Az}_n(\text{Spec } \hat{\mathcal{O}}_{C,x} \times_C W)$  correspond to  $\hat{\mathcal{A}}$ .

The functor

$$\text{Sch}/C : D \mapsto \text{Az}_n(D \times_C W)$$

is locally finitely presented in the sense that its restriction to affine  $k$ -schemes commutes with filtered direct limits. It follows from the Artin approximation theorem [1, Cor (2.2)] that there exists  $x \rightarrow C' \xrightarrow{\text{etale}} C$  and  $\mathcal{A}' \in \text{Az}_n(C' \times_C W)$  such that formally

$$“\mathcal{A}' \equiv \tilde{\mathcal{A}} \text{ mod } m_x''$$

which means

$$\mathcal{A}' \otimes_{\mathcal{O}_{C'}} \mathcal{O}_x \cong \tilde{\mathcal{A}} \otimes_{\hat{\mathcal{O}}_{C,x}} \mathcal{O}_x \cong \mathcal{A}_0$$

finishing the proof.  $\square$

## 10. CREATING A FAMILY

We prove the following result,

**Proposition 10.1.** *Let  $X/k$  a smooth, proper, connected surface and let  $\alpha \in \text{Br}(X)_n$ . Then there exists a smooth connected  $k$ -variety  $W$  of dimension 3 and morphisms*

$$\begin{array}{ccc} W & \xrightarrow{g} & X \\ f \downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

such that

- (1)  $f$  is smooth.
- (2) The generic fiber of  $f$  is geometrically connected.
- (3)  $f$  is proper over a neighborhood of 0.
- (4)  $Y \stackrel{\text{def}}{=} W_0 = f^{-1}(0)$  splits  $\alpha$ .
- (5)  $W_1 = f^{-1}(1) \neq \emptyset$  and  $g : W_1 \rightarrow X$  is an open immersion.

*Proof of Proposition 10.1.* We know that  $\alpha_{k(X)}$  is represented by a division algebra. Hence by lemma 5.8 we know there is an Azumaya algebra  $\mathcal{A} \in \text{Az}_m(X)$  on  $X$  representing  $\alpha$  which is generically a division algebra. We choose a line bundle  $\mathcal{L}$  and a section  $s \in \mathcal{A} \otimes_X \mathcal{L}$  so that  $Y_s$  is smooth and splits  $\mathcal{A}$ . Put

$$\mathcal{B} = \oplus_n \mathcal{L}^{-n} T^n$$

Then  $Y_s$  is defined by the locally principal  $\mathcal{B}$  ideal generated by  $P(T)\mathcal{L}^{-m}$  where  $P(T)$  is the reduced characteristic polynomial of  $s$ .

Choose  $w_1, \dots, w_m$  distinct global sections of  $\mathcal{L}$  and put

$$Q(T) = (T - w_1) \cdots (T - w_m)$$

and

$$R(t, T) = (1 - t)P(T) + tQ(T)$$

We view  $R(t, T)$  as a section of

$$\mathcal{B}[t] \otimes_X \mathcal{L}^m$$

We let  $W' \subset L \times \mathbb{A}^1$  be defined by the locally principal  $\mathcal{B}[t]$  ideal generated by  $R(t, T)\mathcal{L}^{-m}$ . Thus we have the following diagram

$$\begin{array}{ccccc} W' & \xrightarrow{\text{pr}_1} & L & \longrightarrow & X \\ \text{pr}_2 \downarrow & & & & \\ \mathbb{A}^1 & & & & \end{array}$$

We denote the composition of the horizontal arrows by  $g'$  and the vertical arrow by  $f'$ . We check the following things.

- (1)  $W' \rightarrow X \times \mathbb{A}^1$  is finite flat. To see this we note that  $R(t, T)$  is of the form  $T^m +$  lower terms in  $T$ . Thus  $\mathcal{B}[t]/(R(t, T)\mathcal{L}^{-m})$  is locally free of rank  $m$  over  $X \times \mathbb{A}^1$ . For use below we record  $W' = \underline{\text{Spec}} \mathcal{C}$  where

$$\mathcal{C} = \left( \bigoplus_{i=0}^m \mathcal{L}^{-i} T^i \right) [t]$$

It follows in particular that  $f' : W' \rightarrow \mathbb{A}^1$  is proper and flat.

- (2) The fiber at 0 of  $f' : W' \rightarrow \mathbb{A}^1$  is equal to  $Y_s$ . This is clear since this fiber is defined by  $P(T)$ .
- (3) The fiber at 1 of  $f' : W' \rightarrow \mathbb{A}^1$  is defined by  $Q(T)$  and can be written as  $X = X_1 \cup \dots \cup X_m$  where  $g'$  restricted to each  $X_i$  defines an isomorphism  $X_i \cong X$ . The singular points of  $(f')^{-1}(1)$  occur at the intersections  $X_i \cap X_j$ . This happens at the points of  $X$  where the sections  $w_i, w_j$  are equal.

Now put

$$W'' = W' - X_2 \cup \dots \cup X_m$$

Then the fibers of  $f'|W''$  are smooth at 0, 1. Since smoothness is an open condition on flat maps [14, Thm 17.5.1] there is a neighborhood  $U$  of 0, 1 such that  $(f')^{-1}(U) \rightarrow U$  is smooth. And of course  $(f')^{-1}(U - 1) \rightarrow (U - 1)$  is still proper. We let  $W$  be the smooth locus of  $f' | W''$ . Then  $W$  contains  $(f')^{-1}(U)$ . We let  $f, g$  be the restrictions of  $f', g'$  to  $W$ .

The only property that remains to be proved is that the generic fiber of  $f$  is geometrically connected. It is clear that  $f$  and  $f'$  have the same generic fiber hence we consider  $f'$  which is proper.

Consider the Stein factorization [15] for proper morphisms.

$$W' \xrightarrow{p} \text{Spec } \Gamma(W', \mathcal{O}_{W'}) \xrightarrow{q} \mathbb{A}^1$$

Here  $p$  has geometrically connected fibers [12, Remarque 4.3.4] and  $q$  is finite. We need to show that  $q$  is an isomorphism. I.e. that  $\Gamma(W', \mathcal{O}_{W'}) = k[t]$ . This follows from the fact that

$$\Gamma(W', \mathcal{O}_{W'}) = \Gamma(X, \mathcal{C}) = k[t] \quad \square$$

## 11. THE UNRAMIFIED CASE

**Theorem 11.1.** *Let  $X$  be a projective smooth connected surface over algebraically closed field. Assume that  $\alpha \in \text{Br}(X) \subset \text{Br}(k(X))$  has period  $n$ . Then the index of  $\alpha$  is  $n$ .*

*Proof.* We construct

$$\begin{array}{ccccc} Y & \longrightarrow & W & \xrightarrow{g} & X \\ \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & \mathbb{A}^1 & & \end{array}$$

as in Proposition 10.1.

Choose  $\eta \in H^2(X, \mu_n)$  whose image is  $\alpha$ . From the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}(Y)/n \text{Pic}(Y) & \longrightarrow & H^2(Y, \mu_n) & \longrightarrow & \text{Br}(Y)_n \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \text{Pic}(X)/n \text{Pic}(X) & \longrightarrow & H^2(X, \mu_n) & \longrightarrow & \text{Br}(X)_n \longrightarrow 0 \end{array}$$

we see that we can choose  $\mathcal{L} \in \text{Pic}(Y)$  such that the image of  $\mathcal{L}$  in  $H^2(Y, \mu_n)$  is  $-\eta_Y$ . Consider the following Azumaya algebra on  $Y$

$$\mathcal{B}_0 = \mathcal{E}nd(\mathcal{L}_Y \oplus \mathcal{O}_Y^{n-1})$$

By Lemma 4.2 we find  $\text{cl}(\mathcal{B}_0) = \eta_Y$ . Using Lemma 8.4 we may find an elementary transform  $\mathcal{A}_0$  of  $\mathcal{B}_0$  such that  $H^2(Y, \mathcal{A}_0/\mathcal{O}_Y) = 0$  and such that  $\text{cl}(\mathcal{A}_0) = \text{cl}(\mathcal{B}_0) = \eta_Y$ . According to Theorem 9.1 there exists an etale neighborhood  $C' \rightarrow \mathbb{A}^1$  of 0 such that  $f' : W' = C' \times_{\mathbb{A}^1} W \rightarrow C'$  is proper and smooth and an Azumaya algebra  $\mathcal{A}$  on  $W' = C' \times_{\mathbb{A}^1} W$  such that  $\mathcal{A}_Y = \mathcal{A}_0$ . Since  $\text{cl}(-)$  is compatible with base change we have  $\text{cl}(\mathcal{A})_Y = \text{cl}(\mathcal{A}_Y) = \text{cl}(\mathcal{A}_0) = \eta_Y$ .

We have  $\text{cl}(\mathcal{A}) \in H^2(W', \mu_n)$ . According to [18, Prop 1.13] we have

$$(R^2 f'_* \mu_n)_0 = \text{inj lim } H^2(C'' \times_{C'} W', \mu_n)$$

where  $C''$  runs through all etale neighborhoods of  $0 \in C'$ . By proper base change [11, IV-1] we have

$$(R^2 f'_* \mu_n)_0 = H^2(Y, \mu_n)$$

Combining these two results we get

$$H^2(Y, \mu_n) = \text{inj lim } H^2(C'' \times_{C'} W', \mu_n)$$

Now  $\text{cl}(\mathcal{A})$  and  $\eta_{W'}$  have the same image in  $H^2(Y, \mu_n)$ . It follows there is some etale neighborhood  $C'' \rightarrow C'$  of  $0 \in C'$  such that  $\text{cl}(\mathcal{A})_{W''} = \eta_{W''}$  where  $W'' = C'' \times_{C'} W'$ . We replace  $C''$  by  $C'$ ,  $W'$  by  $W''$ ,  $\mathcal{A}$  by  $\mathcal{A}_{W''}$ . We have now arrived at the situation where  $\text{cl}(\mathcal{A}) = \eta_{W'}$ . Going to Brauer groups we find

$$[\mathcal{A}] = [\text{cl}(\mathcal{A})] = [\eta_{W'}] = [\eta]_{W'} = \alpha_{W'}$$

Thus we have shown that the image of  $\alpha \in \text{Br}(k(X))$  in  $\text{Br}(k(W'))$  has index  $n$ . By Theorem 3.1 it is now sufficient to construct a discrete valuation ring in  $k(W')$  with residue field  $k(X)$ .

We proceed as follows. We may extend  $C' \rightarrow \mathbb{A}^1$  to finite morphism of smooth (affine) curves  $C \rightarrow \mathbb{A}^1$ . Then  $\tilde{W} = C \times_{\mathbb{A}^1} W$  is smooth over  $C$  and hence regular. It contains  $W'$  as an open subset and hence  $k(W') = k(\tilde{W})$ .

Since  $k$  is algebraically closed there exists a point  $c \in C$  lying above  $1 \in \mathbb{A}^1$ . Then  $c \times_{\mathbb{A}^1} W$  is a divisor  $D$  in  $\tilde{W}$  which is isomorphic to  $0 \times_{\mathbb{A}^1} W$  which is an open subset of  $X$ . The discrete valuation ring in  $k(\tilde{W})$  defined by  $D$  has residue field  $k(X)$  which is what we want.  $\square$

## 12. THE RAMIFIED CASE

**Theorem 12.1.** *Let  $k$  be an algebraically closed field. Assume that  $X/k$  is a smooth projective connected surface. Let  $\alpha \in \text{Br}(k(X))$  be of period  $n$ , prime to the characteristic of  $k$ . Then the index of  $\alpha$  is  $n$ .*

*Proof.* It is standard that we may reduce to the case that  $\text{period}(\alpha)$  is prime  $l$ , different from the characteristic (we could already have done this earlier but for the unramified case does not simplify the proof). To avoid trivialities we assume  $\alpha \neq 0$ .

To start the proof we construct morphisms

$$\begin{array}{ccc} W & \xrightarrow{g} & X \\ f \downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

of smooth connected varieties with  $\dim W = 3$  and with the following properties.

- (1) The morphism  $f : W \rightarrow \mathbb{A}^1$  is smooth.
- (2) The generic geometric fiber  $W_{\bar{\eta}}$  of  $f : W \rightarrow \mathbb{A}_k^1$  is projective and connected.
- (3) The fiber  $W_1$  of  $f$  in  $1 \in \mathbb{A}_k^1$  is non-empty and the restriction of  $g : W \rightarrow X$  to  $W_1$  is birational.
- (4) The inverse image  $r^*(\alpha)$  of  $\alpha$  through the composition  $r : W_{\bar{\eta}} \rightarrow W_{\eta} \rightarrow W \rightarrow X$  is unramified on the surface  $W_{\bar{\eta}}$ .

Assume we have constructed such morphisms. By the unramified we know that  $(\alpha)_{k(W_{\bar{\eta}})}$  has index 1 or  $l$ . We can pull this back to a finite extension of  $k(\mathbb{A}^1)$ . Hence there exists a finite extension  $L/k(\mathbb{A}^1)$  such that the extension of  $\alpha$  to  $k(W_{\eta} \times_{k(\mathbb{A}^1)} L)$  has index 1 or  $l$ . Now  $L$  is the function field of a smooth curve  $C$  such that there is a finite map  $C \rightarrow \mathbb{A}^1$ . Hence  $k(W_{\eta} \times_{k(\mathbb{A}^1)} L)$  is the function field of  $W' = C \times_{\mathbb{A}^1} W$ . So we have shown that the image of  $\alpha \in \text{Br}(k(X))$  in  $k(W')$  has index 1 or  $l$ . So what is left to do is to construct a discrete valuation on  $k(W')$  with residue field  $k(W)$ . To do this we take a point  $c \in C$  lying above  $0 \in \mathbb{A}^1$ . Then the discrete valuation we want is the one associated to the divisor  $c \times_{\mathbb{A}^1} W$ .

So what remains to be done is to construct the morphisms as indicated in the diagram.

Let  $D \subset X$  be the ramification divisor of  $\alpha$ . By resolution of singularities we may assume that  $D$  has normal crossings. We claim that we can find smooth effective divisors  $E, E'$  such that  $D + E$  has normal crossings and such that

$$D + E \sim l(D + E')$$

We choose an ample line bundle  $\mathcal{M}$  on  $X$ . For  $r$  sufficiently big we have that  $\mathcal{O}_X((l-1)D) \otimes \mathcal{M}^{rl}$  and  $\mathcal{M}^r$  are very ample. Let  $E'$  be the zeroes of a generic section of  $\mathcal{M}^r$ . Then  $E'$  is smooth by Bertini. Let  $E$  be the zeroes of a generic section of  $\mathcal{O}_X((l-1)D) \otimes \mathcal{M}^{rl}$ . Then  $E$  is smooth and  $D \cap E$  is smooth as well by Bertini (thus  $D + E$  has normal crossings). In the last application of Bertini we have used that a generic hyperplane section will miss the finite number of singular points of  $D \cap E$  [15, II.8.18.1]. We now have isomorphisms

$$\begin{aligned} \mathcal{O}_X(D + E') &\cong \mathcal{O}_X(D) \otimes \mathcal{M}^r \stackrel{\text{def}}{=} \mathcal{L} \\ \mathcal{O}_X(D + E) &\cong \mathcal{O}_X(lD) \otimes \mathcal{M}^{lr} = \mathcal{L}^l \end{aligned}$$

Taking the images of 1 under both isomorphisms we get sections  $s_1 \in H^0(X, \mathcal{L})$ ,  $s_0 \in H^0(X, \mathcal{L}^l)$  with zeroes respectively  $D + E'$  and  $D + E$ . Note that  $s_1$  and  $s_0$  are both sections of  $H^0(X, \mathcal{L}^l)$ .

As before we define

$$\begin{aligned} \mathcal{B} &= \bigoplus_{i \geq 0} \mathcal{L}^{-i} T^i \\ L &= \underline{\text{Spec}} \mathcal{B} \end{aligned}$$



of  $A_{l-1}$  singularities. Using  $\lceil \frac{l-1}{2} \rceil$  blowups of the closure of the singular locus we may simultaneously resolve the fibers of  $f^{-1}(U) \rightarrow U$ . We end up with a probably singular but integral threefold  $\tilde{f}: \tilde{W}' \rightarrow W'$  such that  $\tilde{f}^{-1}(U) \rightarrow U$  is smooth.

Now note that the ideals we have blown up define subvarieties of dimension 1. Since all components of  $(f')^{-1}(1)$  are of dimension 2 this means that  $\tilde{f}^{-1}(1)$  will consist of the strict transform of  $(f')^{-1}(1)$  plus possibly some exceptional components.

Now let  $W''$  be obtained from  $\tilde{W}'$  by removing from the fiber of 1 all components, except the strict transform of one components of  $W_1$ . Finally let  $W$  be the smooth locus of  $W''$ . Then  $W$  has all desired properties except that we still need to show that  $W_{\bar{\eta}}$  splits  $\alpha$ .

We have the diagram of maps

$$W_{\bar{\eta}} = \tilde{W}'_{\bar{\eta}} \rightarrow W'_{\bar{\eta}} \rightarrow (X \times \mathbb{A}^1)_{\bar{\eta}}$$

Here  $W'_{\bar{\eta}}/(X \times \mathbb{A}^1)_{\bar{\eta}}$  is a cyclic covering which is ramified in a normal crossing divisor with smooth components which contains the ramification locus of  $\alpha$  and  $\tilde{W}'_{\bar{\eta}}$  is the canonical desingularization of  $W'_{\bar{\eta}}$  obtained by repeatedly blowing up the singular locus. It now suffices to invoke Proposition 5.7.  $\square$

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